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Anti Eccentric Harmonic Index of Graphs

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Abstract: In this paper, we introduce a new index for the connected simple graph G namely, anti eccentric harmonic index $H'_e(G)$, defined by, $\frac{2}{e'(u)+e'(v)}$ for all edges uv in G, where $e'(v) = \nu - e(v)$. We determine the exact values of $H'_e(G)$ for some class of graphs and also we found the bound for $H'_e(G)$ where G is a graph of order and size n and m, respectively.

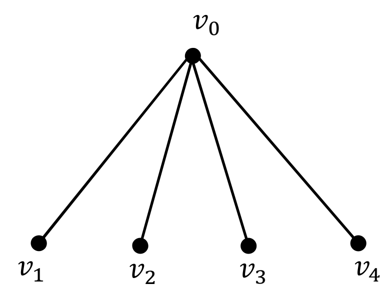
Keywords: harmonic index, eccentric harmonic index, anti harmonic index, anti eccentric harmonic index.

MSC : 05C07, 05C12, 05C09.

1. Introduction

In this paper, $G = (V, E)$ be simple finite undirected connected graph with vertex set V and edge set E. Let $|V(G)| = \nu$ and $|E(G)| = \epsilon$, be the order and size of the graph G, respectively. For every vertex $v \in V$, the open neighbourhood $N(v)$, is the set of all vertices that are adjacent to v. The degree of vertex v in a graph G is $d(v) = |N(v)|$. A vertex of degree 1 is called pendant vertex.

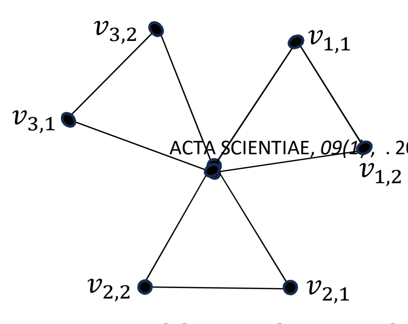
The cartesian product of two graphs G and H is denoted as G□H with the vertex set $V(G\Box H) = V(G) \times V(H)$ in which (u, v) is adjacent to (u', v'), if and only if $u = u'$ and $vv' \in E(H)$ or $uu' \in E(G)$ and $v = v'$. A complete graph is one in which every pair of vertices connected by an edge. A complete graph with m vertices is denoted as K_m. For $m \geq 2$, S_m denotes the star on m vertices in which one vertex is adjacent to all other remaining $m - 1$ vertices. Also $S_m \cong K_{1,m-1}$, see Figure 1.1.



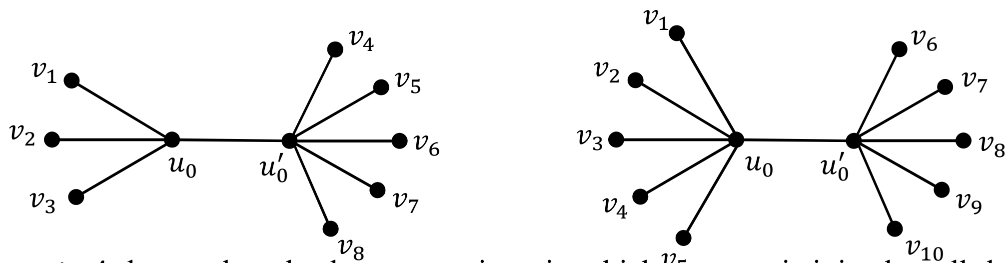
$S_5 -$ Star on 5 vertices.

Figure 1.1

The friendship graph F_n is obtained by joining n copies of the C_3 with a common vertex, which is also known as universal vertex for the graph F_n, see Figure 1.2 for F_3.



In [1] double star and bistar is defined. A double star is a graph obtained by joining an edge to center of $K_{1,n}$ and $K_{1,m}$, for $n > m \geq 2$ and it is denoted by $B_{n,m}$, see Figure 1.3 (a) for $B_{3,5}$. If $n = m$ in the double star it is called as bistar and is denoted by $B_{n,n}$, see Figure 1.3 (b) for $B_{5,5}$.



Double star $B_{3,5}$

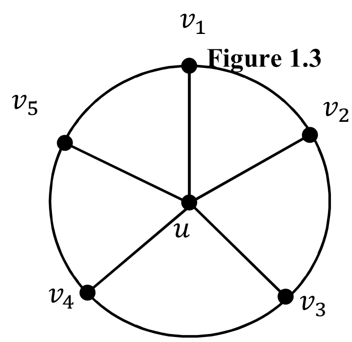
(a)

Bistar $B_{5,5}$

(b)

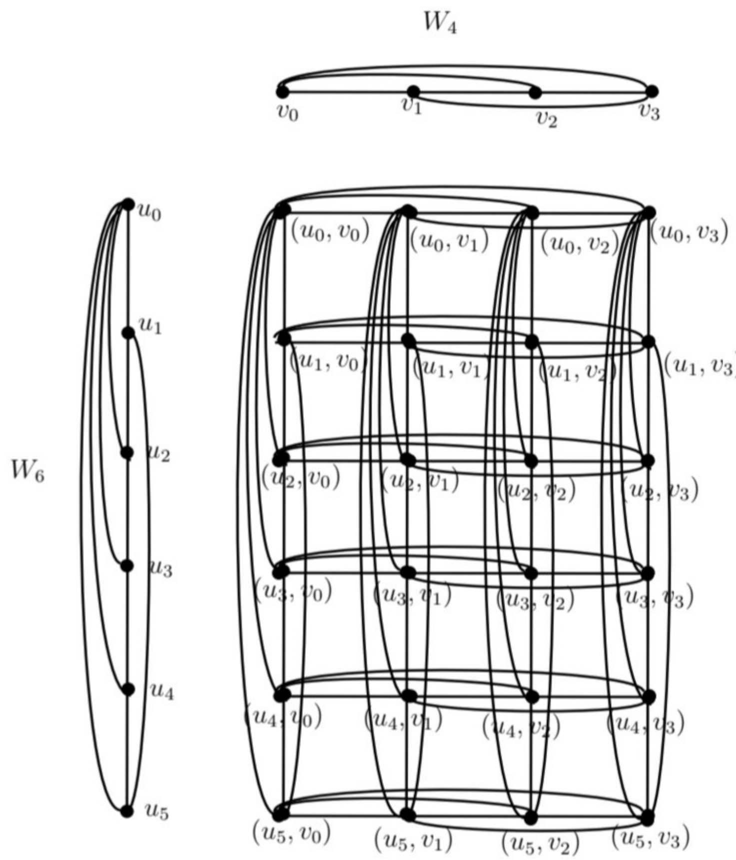
Figure 1.3

A W_m $m \geq 4$ denote the wheel on m vertices, in which a vertex is joined to all the vertices of a cycle C_{m-1}, see Figure 1.4.



W_6- Wheel on 6 vertices.

Figure 1.4



$W_6 \square W_4$

Figure 1.5

A topological index is a numerical quantity related to a graph. In a theoretical chemistry, topological indices is called as molecular descriptor. In [2], Fajtlowicz introduced an another topological index named harmonic index, given by

$$H(G) = \sum_{uv \in E(G)} \frac{2}{d(u) + d(v)}.$$

In [3] Sowaity et.al., had introduced the eccentric harmonic index as similar to harmonic index, by considering the eccentricity of the vertices instead of degree of the vertices, which is given by,

$$H_e(G) = \sum_{uv \in E(G)} \frac{2}{e(u) + e(v)}.$$

In [4] Sivasankar and Haritha had introduced a topological index named anti harmonic index given by

$$H'(G) = \sum_{uv \in E(G)} \frac{2}{d'(u) + d'(v)}.$$

Various indices are studied by [5, 6, 7, 8, 9, 10, 11, 12]. In similarity with anti harmonic index, we introduce anti eccentric harmonic index $H'_e(G)$.

2. Anti Eccentric Harmonic Index for some Graphs

In [4] Sivasankar and Haritha introduced the anti degree of a vertex v in G as $d'_G(v) = \nu - d_G(v)$. In this direction we may think of anti eccentricity of a vertex. So we define anti eccentricity of a vertex v for a connected graph G of order ν as $e'_G(v) = \nu - e_G(v)$. For any connected graph G, $1 \le e_G(v) \le \nu - 1$ and $1 \le e'_G(v) \le \nu - 1$. In help of anti eccentricity of a vertex we introduce anti eccentric harmonic index $H'_e(G)$ of a graph G similar to anti harmonic index $H'(G)$ of a graph G and we compute the anti eccentric harmonic index for some well-known graphs.

Definition 2.1. The anti eccentric harmonic index $H'_e(G)$ of a graph G is defined as

$$H'_e(G) = \sum_{uv \in E(G)} \frac{2}{e'(u) + e'(v)}.$$

Theorem 2.2. For $n \ge 2$, the anti eccentric harmonic index of star S_n is

$$H'_e(S_n) = \begin{cases} 1 & : n = 2 \\ \frac{2(n-1)}{2n-3} & : n \ge 3 \end{cases}$$

Proof. The anti eccentric harmonic index of S_2 is 1. Let $V(S_n) = \{v_0, v_1, \dots, v_{n-1}\}$. For $n \ge 3$, the anti eccentric harmonic index will be

$$H'_e(S_n) = \sum_{i=1}^{n-1} \frac{2}{e'(v_0) + e'(v_i)} = \frac{2(n-1)}{2n-3}.$$

□

Theorem 2.3. For $n \ge 2$, the anti eccentric harmonic index of path P_n is

$$H'_e(P_n) = \begin{cases} 4\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{2i+1} & : n \text{ is odd} \\ 2\left(\sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} \frac{2}{2i+1} + \frac{1}{n}\right) & : n\ is\ even \end{cases}$$

Proof. Let $V(P_n) = \{v_1, v_2, \dots, v_n\}$, the anti eccentric harmonic index of P_n is

Case 1: If n is odd

$$H'_e(P_n) = \sum_{i=1}^{n-1} \frac{2}{e'(v_i) + e'(v_{i+1})} = 2\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \frac{2}{i+i+1} = 4\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{2i+1}.$$

Case 2: If n is even

$$H'_e(P_n) = \sum_{\substack{i=1\\ i\neq\frac{n}{2}}}^{n-1} \frac{2}{e'(v_i)+e'(v_{i+1})} + \frac{2}{e'\left(v_{\frac{n}{2}}\right)+e'\left(v_{\frac{n}{2}+1}\right)},$$

$$= 2\sum_{i=1}^{\lfloor\frac{n-1}{2}\rfloor} \frac{2}{i+i+1} + \frac{2}{\frac{n}{2}+\frac{n}{2}}.$$

$$= 2\left(\sum_{i=1}^{\lfloor\frac{n-1}{2}\rfloor} \frac{2}{2i+1} + \frac{1}{n}\right).$$

□

Theorem 2.4. For $m, n \geq 2$, the anti eccentric harmonic index of a double star graph $B_{m,n}$ is

$$H'_e(B_{m,n}) = \frac{2(m+n)}{2(m+n)-1} + \frac{1}{m+n}.$$

Proof. Let $V(B_{m,n}) = \{u_0, u'_0, v_1, v_2, \dots, v_m, v_{m+1}, \dots, v_{m+n}\}$. Then,

$$H'_e(B_{m,n}) = \sum_{i=1}^{m} \frac{2}{e'(u_0)+e'(v_i)} + \sum_{i=m+1}^{m+n} \frac{2}{e'(u'_0)+e'(v_i)} + \frac{2}{e'(u_0)+e'(u'_0)}.$$

$$= m\left(\frac{2}{(m+n+2-3)+(m+n+2-2)}\right)$$

$$+n\left(\frac{2}{(m+n+2-3)+(m+n+2-2)}\right)$$

$$\frac{2}{(m+n+2-2)+(m+n+2-2)}.$$

$$= \frac{2(m+n)}{2(m+n)-1} + \frac{1}{m+n}.$$

□

Remark 2.1. In above Theorem 2.4 if we assume $m = n$, then the graph becomes a bistar graph. For which anti eccentric harmonic index is

$$H'_e(B_{m,n}) = \frac{4\mathrm{m}}{4m-1} + \frac{1}{2m}.$$

Theorem 2.5. For $n \geq 4$, the anti eccentric harmonic index of wheel W_n is

$$H'_e(W_n) = \begin{cases} 2 & : n = 4 \\ \frac{\mathrm{n}-1}{n-2} + \frac{2(n-1)}{2n-3} & : n \geq 5 \end{cases}$$

Proof. Let $V(W_n) = \{v_0, v_1, \dots, v_{n-1}\}$. $H'_e(W_n) = 2$, $n = 4$ is obvious and the anti eccentric harmonic index of wheel graph for $n \geq 5$ is given by

$$H'_e(W_n) = \sum_{i=1}^{n-1} \frac{2}{e'(v_i) + e'(v_{i+1})} + \sum_{i=1}^{n-1} \frac{2}{e'(v_0) + e'(v_i)},$$

where the addition $i + 1$ in suffix of v is taken modulo $n - 1$.

$$= (n-1)\left(\frac{2}{n-2+n-2}\right) + (n-1)\left(\frac{2}{n-1+n-2}\right).$$

$$= \frac{n-1}{n-2} + \frac{2(n-1)}{2n-3}.$$

□

Theorem 2.6. The anti eccentric harmonic index of a self-centered graph G of order ν and size ϵ with eccentricity as k is

$$H'_e(G) = \frac{\epsilon}{\nu - k}.$$

Proof. In a self-centered graph, $e(v) = k$, for every $v \in V(G)$ and $e'(v) = \nu - k$. Then,

$$H'_e(G) = \sum_{uv \in E(G)} \frac{2}{e'(u) + e'(v)} = \frac{\epsilon}{\nu - k}.$$

□

Remark 2.2. By using above Theorem 2.6 anti eccentric harmonic index of some self-centered graph are given by

Graph G	**ν**	ϵ	k**-(eccentricity of G)**	$H'_e(G)$
K_n	n	$\binom{n}{2}$	1	$\frac{n}{2}$
C_n	n	n	$\frac{n}{2}$ if n is even and $\frac{n-1}{2}$ if n is odd	$\begin{cases} 2 & : if\ n\ is\ even \\ \frac{2n}{n+1} & : if\ n\ is\ odd \end{cases}$
$K_{m,n}$	$m+n$	mn	2	$\begin{cases} \frac{n^2}{2(n-1)} & : if\ m=n \\ \frac{mn}{m+n-2} & : if\ m \neq n \end{cases}$
$C_m \square C_n$	mn	$2mn$	$\frac{m}{2}$ if m is even and $\frac{m-1}{2}$ if m is odd	$\begin{cases} \frac{4mn}{2\mathrm{mn} - (\mathrm{m}+\mathrm{n})} & : if\ m\ and\ n\ are\ even \\ \frac{4\mathrm{mn}}{2\mathrm{mn} - (\mathrm{m}+\mathrm{n}-2)} & : if\ m\ and\ n\ are\ odd \\ \frac{4\mathrm{mn}}{2\mathrm{mn} - (\mathrm{m}+\mathrm{n}-1)} & : if\ m\ is\ odd\ and\ n\ is\ even \end{cases}$
$K_m \square K_n$	mn	$m\binom{n}{2} + n\binom{m}{2}$	2	$\frac{mn(m+n-2)}{2(mn-2)}$
$C_m \square P_2$	$2m$	$3m$	$\frac{m}{2}+1$ if m is even and $\frac{m+1}{2}$ if m is odd	$\begin{cases} \frac{6m}{3m-2} & : if\ m\ is\ even \\ \frac{6m}{3m-1} & : if\ m\ is\ odd \end{cases}$
$K_m \square P_2$	$2m$	m^2	2	$\frac{m^2}{2(m-1)}$
$G \square H$	$\nu(G)\nu(H)$	$\nu(G)\epsilon(H) + \nu(H)\epsilon(G)$	$k+\ell$	$\frac{\nu(G)\epsilon(H)+\nu(H)\epsilon(G)}{[\nu(G)\nu(H)]-[k+\ell]}$

Petersen Graph	10	15	2	$\frac{15}{8}$

Theorem 2.7. The anti eccentric harmonic index of friendship graph F_n, for $n \geq 1$ is

$$H'_e(F_n) = \frac{n}{2n-1} + \frac{4n}{4n-1}.$$

Proof. Let $V(F_n) = \{v_0\} \cup \{v_{i,1}, v_{i,2} \mid i = 1, 2, \ldots, n\}$ see Figure 1.2. The anti eccentric harmonic index of friendship graph F_n is given as

$$H'_e(F_n) = \sum_{i=1}^{n} \frac{2}{e'(v_{i,1}) + e'(v_{i,2})} + \sum_{i=1}^{n} \sum_{j \in \{1,2\}} \frac{2}{e'(v_0) + e'(v_{i,j})}.$$

$$= n\left(\frac{2}{2n+1-2+2n+1-2}\right) + 2n\left(\frac{2}{2n+1-2+2n+1-1}\right).$$

$$= \frac{n}{2n-1} + \frac{4n}{4n-1}.$$

□

Theorem 2.8. For $m, n \geq 4$,

$$H'_e(W_m \square W_n) = \begin{cases} \frac{24}{7} & : if\ m = n = 4 \\ \frac{3}{2m-1} + \frac{8(m-1)}{8m-5} + \frac{10(m-1)}{4m-3} & : if\ m \geq 5\ and\ n = 4 \\ \frac{3}{2n-1} + \frac{8(n-1)}{8n-5} + \frac{10(n-1)}{4n-3} & : if\ m = 4\ and\ n \geq 5 \\ (m+n-2)\left(\frac{2}{2mn-5} + \frac{1}{mn-3}\right) & \\ +2(m-1)(n-1)\left(\frac{2}{2mn-7} + \frac{1}{mn-4}\right) & : otherwise\ m \geq 5 \\ & \quad and\ n \geq 5 \end{cases}$$

Proof.
Case 1. If $m = 4$ and $n = 4$. The graph $W_4 \square W_4$ is a self-centered graph. Then from Theorem 2.6, the anti eccentric harmonic index will be

$$H'_e(W_4 \square W_4) = \frac{24}{7}.$$

Case 2. If $m \geq 5$ and $n = 4$. Let $V(W_m) = \{u_0, u_1, \dots, u_{m-1}\}$ and $V(W_4) = \{v_0, v_1, v_2, v_3\}$ then $V(W_m \square W_4) = \{(u_i, v_j) \mid i = 0, 1, \dots, m-1 \text{ and } j = 0, 1, 2, 3\}$ and $| W_m \square W_4 | = 4m$. As the eccentricity of vertices in W_m is either 1 or 2 and the eccenticity of vertices in W_4 is being 1. In $W_m \square W_4$, the eccentricity of each vertex will be either 2 or 3 only, as given below, (see Figure 1.5, for $W_6 \square W_4$).

$$e(u_0, v_j) = 2 \text{ for } j = 0,1,2,3.$$

$$e(u_i, v_j) = 3 \text{ for } i = 1, 2, \dots, m-1 \text{ and } j = 0, 1, 2, 3.$$

The edges of $W_m \square W_4$ have their eccentricity for its end point are only $\{2,2\}, \{2,3\}$ and $\{3,3\}$. The number of edges with their eccentricities $\{2,2\}, \{2,3\}$ and $\{3,3\}$ are only $6, 4(\mathrm{m}-1)$, and $10(\mathrm{m}-1)$, respectively. For $(u, v) \in V(W_m \square W_4), e'(u, v) = 4m - e(u, v)$. The anti eccentric harmonic index of $W_m \square W_4$ is given by

$$H'_e(W_m \square W_4) = 6\left(\frac{2}{4m-2+4m-2}\right) + 4(m-1)\left(\frac{2}{4m-2+4m-3}\right) + 10(m-1)\left(\frac{2}{4m-3+4m-3}\right).$$

Thus,

$$H'_e(W_m \square W_4) = \frac{3}{2m-1} + \frac{8(m-1)}{8m-5} + \frac{10(m-1)}{4m-3}.$$

Case 3. If $m = 4$ and $n \geq 5$. As $W_m \square W_n \cong W_n \square W_m$, it is immediate, by replacing m by n in previous case.

Case 4. If $m \geq 5$ and $n \geq 5$. Let Let $V(W_m) = \{u_0, u_1, \dots, u_{m-1}\}$ and $V(W_4) = \{v_0, v_1, \dots, v_{n-1}\}$ then $V(W_m \square W_n) = \{(u_i, v_j) \mid i = 0, 1, \dots, m-1 \text{ and } j = 0, 1, \dots, n-1\}$ and $| W_m \square W_n | = mn$. As the eccentricity of wheel is either 1 or 2. In $W_m \square W_n$, the eccentricity of each vertex will be 2, 3 or 4 only as given below.

$$e(u_0, v_0) = 2$$

$$e(u_i, v_0) = 3 \text{ for } i = 0, 1, \dots, m-1$$

$$e(u_0, v_j) = 3 \text{ for } j = 0, 1, \dots, n-1$$

$$e(u_i, v_j) = 4 \text{ for } i, j \neq 0.$$

The edges of $W_m \square W_n$ have their eccentricity for its end point are only $\{2,3\}, \{3,3\}, \{3,4\}$ and $\{4,4\}$. The number of edges with their eccentricities $\{2,3\}, \{3,3\}, \{3,4\}$ and $\{4,4\}$ are only $\mathrm{m}+\mathrm{n}-2, \mathrm{m}+$

$\mathrm{n}-2, 2(\mathrm{m}-1)(\mathrm{n}-1)$ and $2(\mathrm{m}-1)(n-1)$, respectively. For $(u,v) \in V(W_m \square W_n), e'(u,v) = mn - e(u,v)$. The anti eccentric harmonic index of $W_m \square W_n$ is given by

$$H'_e(W_m \square W_n) = (m+n-2)\left(\frac{2}{mn-2+mn-3}\right) + (m+n-2)\left(\frac{2}{mn-3+mn-3}\right) + 2(m-1)(n-1)\left(\frac{2}{mn-3+mn-4}\right) + 2(m-1)(n-1)\left(\frac{2}{mn-4+mn-4}\right).$$

Thus,

$$H'_e(W_m \square W_n) = (m+n-2)\left(\frac{2}{2mn-5} + \frac{1}{mn-3}\right) + 2(m-1)(n-1)\left(\frac{2}{2mn-7} + \frac{1}{mn-4}\right).$$

□

Theorem 2.9. For $m \geq 4$, W_m be the wheel graph with order m and P_2 be the path with order 2. Then,

$$H'_e(W_m \square P_2) = \begin{cases} \frac{8}{3} & : m = 4 \\ \frac{1}{2(m-1)} + \frac{4(m-1)}{4m-5} + \frac{3(m-1)}{2m-3} & : m \geq 5 \end{cases}$$

Proof.
Case 1. If $m = 4$, the graph $W_4 \square P_2$ is a self-centered and by Theorem 2.6, $H'_e(W_4 \square P_2) = \frac{8}{3}$.

Case 2. If $m \geq 5$. Let $V(W_m) = \{u_0, u_1, \dots, u_{m-1}\}$ and $V(P_2) = \{v_1, v_2\}$ then $V(W_m \square P_2) = \{(u_i, v_j) \mid i = 0, 1, \dots, m-1 \text{ and } j = 1, 2\}$ and $|W_m \square P_2| = 2m$. As the eccentricity of vertices in wheel is either 1 or 2 and P_2 is 1. In $W_m \square P_2$, the eccentricity of each vertex will be 2 or 3 only, as given below.

$$e(u_0, v_j) = 2 \text{ for } j = 1,2$$

$$e(u_i, v_j) = 3 \text{ for } i = 0, 1, \dots, m-1 \text{ and } j = 1, 2.$$

The edges of $W_m \square P_2$ have their eccentricity for its end point are only $\{2,2\}, \{2,3\}$ and $\{3,3\}$. The number of edges with their eccentricities $\{2,2\}, \{2,3\}$ and $\{3,3\}$ are only $1, 2(\mathrm{m}-1)$ and $3(\mathrm{m}-1)$, respectively. For $(u,v) \in V(W_m \square P_2), e'(u,v) = 2m - e(u,v)$. The anti eccentric harmonic index of$W_m \square P_2$ is given by

$$H'_e(W_m \square P_2) = \frac{2}{2m-2+2m-2} + 2(m-1)\left(\frac{2}{2m-2+2m-3}\right) + 3(m-1)\left(\frac{2}{2m-3+2m-3}\right).$$

Thus,

$$H'_e(W_m \square P_2) = \frac{1}{2(m-1)} + \frac{4(m-1)}{4m-5} + \frac{3(m-1)}{2m-3}.$$

□

3. Bounds of Anti Eccentric Harmonic Index

Theorem 3.1. Let G be a graph with order n and size m. Then

$$\frac{m}{n-r} \leq H'_e(G) \leq \frac{m}{n-diam(G)},$$

equality holds if and only if G is self-centered graph.

Proof. Let G be graph with n vertices and m edges, for $v_i v_j \in E(G)$

$$2\big(n-diam(G)\big) \leq n-e(v_i)+n-e\big(v_j\big) \leq 2(n-r).$$

So that,

$$\frac{1}{2(n-r)} \leq \frac{1}{n-e(v_i)+n-e\big(v_j\big)} \leq \frac{1}{2\big(n-diam(G)\big)}.$$

Taking summation over all edges of the graph G. We have

$$\frac{m}{2(n-r)} \leq \frac{H'_e(G)}{2} \leq \frac{m}{2\big(n-diam(G)\big)}.$$

$$\frac{m}{n-r} \leq H'_e(G) \leq \frac{m}{n-diam(G)}.$$

The equality holds if $n-e(v_i)+n-e\big(v_j\big) = 2r = 2diam(G)$, which holds if the graph G is self-centered. □

Theorem 3.2. Let G be a graph with order n and size m. Then

$$\frac{m}{n-1} \leq H'_e(G).$$

Proof. Let G be graph with n vertices and m edges, for $v_i v_j \in E(G)$, $\frac{1}{n-1} \leq \frac{2}{n-e(v_i)+n-e(v_j)}$. Taking summation over all edges of G.

$$\frac{m}{n-1} \le H_e^{'}(G).$$

□

Remark 3.1. The bound in Theorem 3.2 is sharp. When G is complete graph, the equality occurs in this inequality given in Theorem 3.2.

Theorem 3.3. Let T be a tree with order n and size m. Then

$$\frac{2m}{2n-3} \le H_e^{'}(T).$$

Proof. Let T be a tree with n vertices and m edges.

Case 1. For $v_i \in V(T)$ such that, $d(v_i) = n-1$. Then the graph will be $T = S_{1,n-1}$. By Theorem 2.2,

$$H_e'(T) = H_e'\left(S_{1,n-1}\right) = \sum_{v_i v_j \in E(T)} \frac{2}{e'(v_i) + e'\left(v_j\right)} = \frac{2m}{2n-3}.$$

Case 2. For $v_i \in V(T)$, such that, $d(v_i) \ne n-1$. Then $e(v) \ne 1$. Therefore $e'(v_i) \ge n-2$, for all $i = 1,2,\dots,n$. Thus,

$$e'(v_i) + e'\left(v_j\right) \le 2(n-2).$$

$$\frac{1}{e'(v_i) + e'\left(v_j\right)} \ge \frac{1}{2(n-2)}.$$

Taking summation over all edges in T.

$$\frac{H_e'(T)}{2} \ge \frac{m}{2(n-2)}.$$

Therefore,

$$\frac{2m}{2n-3} \le \frac{m}{n-2} \le H_e^{'}(T).$$

□

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