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CERTAIN OPERATORS ON FUZZY SOFT SEQUENTIAL TOPOLOGICAL SPACES

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Abstract: The main purpose of this paper is to investigate interior and closure operator in fuzzy soft sequential topological spaces. After introducing and discussing the notions of fuzzy soft sequential open(closed) sets in [4], we define fuzzy soft sequential interior, fuzzy soft sequential closure in fuzzy soft sequential topological spaces and present the properties of these operators with needed examples.

Keywords: fuzzy soft set, fuzzy soft sequential set, fuzzy soft sequential interior, fuzzy soft sequential closure.

1. Introduction

L. A. Zadeh [10] introduced the concept of fuzzy set in 1965, for representing vague concepts. In 1999, **Molodtsov** [7] proposed the concept of soft set theory which is a completely new approach for dealing with uncertainty. In 2001, **Maji et al** [6] introduced the concept of fuzzy soft set to approach vagueness by involving the ideas of both fuzzy set and soft set. **Bose and Indrajit Lahiri** [2] introduced the concept of sequential topological spaces in the year 2002. He proposed properties of sequential set and discussed operators like interior, closure in sequential topology. Fuzzy soft set was further improved by **Ahmad and Kharal** [1] in the year 2009. In 2011, **Tanay and Kandemir** [9] introduced the topological structure of fuzzy soft sets and studied some of its properties with the notions of interior. In 2013, **Atmaca and Zorlutuna** [8] studied the concept of fuzzy soft interior and closure. In continuation, **Mahanta and Das** [5] explored some concepts like interior and closure in fuzzy soft topological spaces in the year 2018. In 2024, we have introduced fuzzy soft sequential set [3] by merging the concept of fuzzy soft set and sequential set. Moreover, we extend the concept into fuzzy soft sequential topological spaces [4] and analyzed in detail.

In this paper, we propose the notion of fuzzy soft sequential interior using fuzzy soft sequential open sets. Further, we introduce the concept of fuzzy soft sequential closure in fuzzy soft sequential topology. We explore the properties, theorems of the above concepts and obtain some related results.

2.Preliminaries

Definition 2.1 [10] A fuzzy set X over a universal set U is a set defined by a function μ_x performing a mapping $\mu_x : U \rightarrow [0,1]$, here this μ_x is the membership function of X, and the value $\mu_x(u)$ will be the grade of membership of u $\in U$.

Definition 2.2 [7] Let U be an initial universe set and E be a set of parameters. A pair (F, E) is called a soft set (over U) if and only if F is a mapping of E into the set of all subsets of the set U.
Definition 2.3 [6] Let $\wp(U)$ denotes the set of all fuzzy sets of U. Let $A_i \subseteq E$, a set of parameters. A pair (F_i, A_i) is called a fuzzy soft set over U, where F_i is a mapping given by $F_i: A_i \to \wp(U)$.
Definition 2.4 [9] Let (γ, X) be an element of $\mathcal{FS}(U; E), \mathcal{P}(\gamma, X)$ be the set of all fuzzy soft subsets of (γ, X) and $\tilde{\tau}$ be a subfamily of $\mathcal{P}(\gamma, X)$. Then $\tilde{\tau}$ is called fuzzy soft topology on (γ, X) if the following conditions are satisfied:

i. $\tilde{\emptyset}_X, (\gamma, X) \in \tilde{\tau}$
ii. $(f, A), (g, B) \in \tilde{\tau} \Rightarrow (f, A) \tilde{\cap} (g, B) \in \tilde{\tau}$
iii. $\{(f, A)_k | k \in K\} \subset \tilde{\tau} \Rightarrow \tilde{\cup}_{k\in K}(f, A)_k \in \tilde{\tau}$.

The pair $(X_\gamma, \tilde{\tau})$ is called a fuzzy soft topological space.
Definition 2.5 [2] Any sequence of subsets of a non void set X is called a sequential set in X. That is, $A(s) = \{A_n\}_{n=1}^{\infty}$, where each A_n is a subset of X, is a sequential set in X. The subsets $A_n, n \in \mathbb{N}$ are called the components of $A(s)$.
Definition 2.6 [2] Let X be a non void set. A collection τ of sequential sets in X is said to form a sequential topology on X if

i. $\emptyset(s), X(s) \in \tau$.
ii. Arbitrary union of members of τ is a member of τ.
iii. Finite intersection of members of τ is a member of τ.

A set X equipped with a sequential topology τ is called a sequential topological space, denoted by (X, τ). The members of τ are called $\tau -$ open sequential sets in X.
Definition 2.7 [3] A sequence of fuzzy soft sets is a mapping from $\mathbb{N}$ to the family of all fuzzy soft sets and is denoted by $\{(F, A)_n\}$ or $\{(F, A)_n; n = 1,2, ...\}$.That is, $\{(F, A)_n, n \in \mathbb{N}\}$ where $(F, A)_n$ for each $n \in \mathbb{N}$ represents components of fuzzy soft set in $\{(F, A)_n\}$ and $n \in \mathbb{N}$, the set of all natural numbers. A sequence of fuzzy soft sets is called fuzzy soft sequential set.
Definition 2.8 [3] The complement of a fuzzy soft sequential set $\{(F, A)_n\}$ is denoted by $\{(F, A)_n\}^c = \{(F, A)_n^c\} = \{(F^c, A)_n\}$, defined by $F_n^c: A \to \mu_x(U)$ is a mapping given by $F_n^c(e_i)(x_j) = 1 - F_n(e_i)(x_j)$ for all $n \in \mathbb{N}$. That is, for each $e_i \in A$ for $i = 1\ to\ m$, and $x_j \in U$ for $j = 1\ to\ n$, $F_n^c(e_i)(x_j)$ is a fuzzy set in U for all $n \in \mathbb{N}$, whose membership function $F_n^c(e_i)(x_j) = 1 - F_n(e_i)(x_j)$ for all $n \in \mathbb{N}$.
Definition 2.9 [3] Union of two fuzzy soft sequential sets $\{(F, A)_n\}$ and $\{(G, B)_n\}$ over U is a fuzzy soft sequential set $\{(H, C)_n\}$, where $C = A \tilde{\cup} B$ and for each $e_i \in C$ for $i = 1\ to\ m$, is defined by

$$H_n(e_i) = \begin{cases} F_n(e_i), & e_i \in A - B \\ G_n(e_i), & e_i \in B - A \\ F_n(e_i) \tilde{\cup} G_n(e_i), & e_i \in A \tilde{\cap} B \end{cases} \quad \text{for all } n \in \mathbb{N}.$$

We denote it by $\{(H, C)_n\} = \{(F, A)_n\} \tilde{\cup} \{(G, B)_n\}$.
Definition 2.10 [3] Intersection of two fuzzy soft sequential sets $\{(F, A)_n\}$ and $\{(G, B)_n\}$ over U is the fuzzy soft sequential set $\{(H, C)_n\}$, where $C = A \tilde{\cap} B$ and for each $e_i \in C$ for $i = 1\ to\ m$, is

defined by $H_n(e_i) = F_n(e_i) \tilde{\cap} G_n(e_i)$ for all $n \in \mathbb{N}$. We denote it by $\{(H,C)_n\} = \{(F,A)_n\} \tilde{\cap} \{(G,B)_n\}$.

Definition 2.11 [4] A family $\tilde{\tau}$ of fuzzy soft sequential sets on U satisfying the properties:

(i) $0^{\mathbb{N}}, 1^{\mathbb{N}} \in \tilde{\tau}$

(ii) $\{(F,A)_n\}, \{(G,B)_n\} \in \tilde{\tau} \Rightarrow \{(F,A)_n\} \tilde{\cap} \{(G,B)_n\} \in \tilde{\tau}$ and

(iii) For any family $\{\{(F_\lambda, A)_n\}, \lambda \in \Lambda\} \in \tilde{\tau} \Rightarrow \tilde{\cup}_{\lambda \in \Lambda} \{(F_\lambda, A)_n\} \in \tilde{\tau}$

is called a fuzzy soft sequential topology on U and the triplet $(U, \tilde{\tau}, E)$ is called a fuzzy soft sequential topological space over U.

Definition 2.12 [4] Let $(U, \tilde{\tau}, E)$ be a fuzzy soft sequential topology on U. Then the members of $\tilde{\tau}$ are called **f**uzzy soft sequential open sets (FSS-open sets). The Complement of a fuzzy soft sequential open set is called a fuzzy soft sequential closed set (FSS -closed set).

3. Main Results

Definition 3.1. Let $(U, \tilde{\tau}, E)$ be a fuzzy soft sequential topological space and $\{(F,A)_n\}$ be a fuzzy soft sequential set over U. Then the union of all fuzzy soft sequential open sets contained in $\{(F,A)_n\}$ is called a **fuzzy soft sequential interior** of $\{(F,A)_n\}$.

In other words, let $\{(F,A)_n\}$ be a fuzzy soft sequential set over U. The **fuzzy soft sequential interior** of $\{(F,A)_n\}$ is denoted by

$$\{(F,A)_n\}^{\circ} = \tilde{\cup} \{\{(G,B)_n\}: \{(G,B)_n\} \tilde{\subseteq} \{(F,A)_n\}, \{(G,B)_n\} \in \tilde{\tau}\}.$$

Remark 3.2. Since the union of all fuzzy soft sequential open sets is a fuzzy soft sequential open set, then the fuzzy soft sequential interior is a fuzzy soft sequential open set.

The example for the definition of fuzzy soft sequential interior is given below.

Example 3.3. Let us consider the universe $U = \{x_1, x_2\}$ and $A, B = \{e_1, e_2\} \subseteq E$ where $\{e_1, e_2\}$ be a collection of sets of parameters. Let $\tilde{\tau} = \{0^{\mathbb{N}}, 1^{\mathbb{N}}, \{(F,A)_n\}\}$ is a fuzzy soft sequential topological space, where the fuzzy soft sequential open set $\{(F,A)_n\}$ is,

$$\{(F,A)_n\} = \left\{ \left(e_1, \left\{ \frac{x_1}{(1/3n)}, \frac{x_2}{(n/n+2)} \right\}\right), \left(e_2, \left\{ \frac{x_1}{(2/5n)}, \frac{x_2}{(1/n)} \right\}\right) \right\} \forall\, n \in \mathbb{N}$$

Let $\{(J,A)_n\}$ be a fuzzy soft sequential set is defined as follows:

$$\{(J,A)_n\} = \left\{ \left(e_1, \left\{ \frac{x_1}{(1/3n-2)}, \frac{x_2}{(n/n+1)} \right\}\right), \left(e_2, \left\{ \frac{x_1}{(\sqrt{3}/2n)}, \frac{x_2}{(1)} \right\}\right) \right\} \forall\, n \in \mathbb{N}$$

Then $\{(J,A)_n\}^{\circ} = \{(0^{\mathbb{N}} \tilde{\cup} \{(F,A)_n\}) \tilde{\subseteq} \{(J,A)_n\}\} = \{(F,A)_n\}$.

Theorem 3.4. Let $(U, \tilde{\tau}, E)$ be a fuzzy soft sequential topological space and $\{(F,A)_n\}$ be a fuzzy soft sequential set over U. Then

i. $\{(F,A)_n\}^{\circ}$ is a fuzzy soft sequential open set and $\{(F,A)_n\}^{\circ}$ is the largest fuzzy soft sequential open set contained in $\{(F,A)_n\}$.

ii. $\{(F,A)_n\}$ is a fuzzy soft sequential open set if and only if $\{(F,A)_n\} = \{(F,A)_n\}^{\circ}$.

Proof.

i. Since $\{(F,A)_n\}^{\circ} = \tilde{\cup} \{\{(G,B)_n\} : \{(G,B)_n\} \tilde{\subseteq} \{(F,A)_n\}, \{(G,B)_n\} \in \tilde{\tau}\}$, $\{(F,A)_n\}^{\circ}$ is itself a fuzzy soft sequential interior of fuzzy soft sequential set $\{(F,A)_n\}$. Then there exists a fuzzy soft sequential open set $\{(G,B)_n\}$ such that $\{(F,A)_n\}^{\circ} \tilde{\subseteq} \{(G,B)_n\} \tilde{\subseteq} \{(F,A)_n\}$ … **[Eqn. 3.1]**

But $\{(G,B)_n\}$ is a fuzzy soft sequential interior of $\{(F,A)_n\}$ implies $\{(G,B)_n\} \tilde{\subseteq} \{(F,A)_n\}^{\circ}$ … **[Eqn. 3.2]**

Using Eqn. 3.1 and Eqn. 3.2, $\{(G,B)_n\} = \{(F,A)_n\}^{\circ}$. Thus $\{(F,A)_n\}^{\circ}$ is a fuzzy soft sequential open set and $\{(F,A)_n\}^{\circ}$ is the largest fuzzy soft sequential open set contained in $\{(F,A)_n\}$.

ii. Assume $\{(F,A)_n\}$ be a fuzzy soft sequential open set. Since $\{(F,A)_n\}^{\circ}$ is a fuzzy soft sequential interior of $\{(F,A)_n\}$, $\{(F,A)_n\}^{\circ} \tilde{\subseteq} \{(F,A)_n\}$ … **[Eqn. 3.3]**

Since $\{(F,A)_n\}^{\circ}$ is the largest fuzzy soft sequential open set contained in $\{(F,A)_n\}$ and by our assumption, $\{(F,A)_n\} \tilde{\subseteq} \{(F,A)_n\}^{\circ}$ … **[Eqn. 3.4]**

By Eqn. 3.3 and Eqn. 3.4, $\{(F,A)_n\} = \{(F,A)_n\}^{\circ}$

On the other hand, let $\{(F,A)_n\} = \{(F,A)_n\}^{\circ}$. By using Remark 3.2, $\{(F,A)_n\}$ is a fuzzy soft sequential open set.

Theorem 3.5. Let $(U,\tilde{\tau},E)$ be a fuzzy soft sequential topological space and $\{(F,A)_n\}, \{(G,B)_n\}$ be the fuzzy soft sequential sets over U. Then the following properties are hold:

i. $(0^{\mathbb{N}})^{\circ} = 0^{\mathbb{N}}$, $(1^{\mathbb{N}})^{\circ} = 1^{\mathbb{N}}$
ii. $\{(F,A)_n\}^{\circ} \tilde{\subseteq} \{(F,A)_n\}$
iii. $(\{(F,A)_n\}^{\circ})^{\circ} = \{(F,A)_n\}^{\circ}$
iv. If $\{(F,A)_n\} \tilde{\subseteq} \{(G,B)_n\}$, then $\{(F,A)_n\}^{\circ} \tilde{\subseteq} \{(G,B)_n\}^{\circ}$
v. $(\{(F,A)_n\} \tilde{\cap} \{(G,B)_n\})^{\circ} = \{(F,A)_n\}^{\circ} \tilde{\cap} \{(G,B)_n\}^{\circ}$
vi. $\{(F,A)_n\}^{\circ} \tilde{\cup} \{(G,B)_n\}^{\circ} \tilde{\subseteq} (\{(F,A)_n\} \tilde{\cup} \{(G,B)_n\})^{\circ}$

Proof:

i. Clearly, union of all fuzzy soft sequential open sets contained in $0^{\mathbb{N}}$ and $1^{\mathbb{N}}$ is always $0^{\mathbb{N}}$ and $1^{\mathbb{N}}$. Hence proved (i).

ii. Using Definition 3.1, $\{(F,A)_n\}^{\circ} \tilde{\subseteq} \{(F,A)_n\}$.

iii. Let $\{(F,A)_n\}^{\circ} = \{(G,B)_n\}$. By (ii) of Theorem 3.4 and Remark 3.2, $\{(G,B)_n\} \in \tilde{\tau}$ if and only if $\{(G,B)_n\}^{\circ} = \{(G,B)_n\}$. Hence $(\{(F,A)_n\}^{\circ})^{\circ} = \{(F,A)_n\}^{\circ}$.

iv. Suppose $\{(F,A)_n\} \tilde{\subseteq} \{(G,B)_n\}$. Let $\{(H,C)_n\} \tilde{\subseteq} \{(F,A)_n\}^{\circ}$. By Definition 3.1 and by (ii), $\{(F,A)_n\}^{\circ} \tilde{\subseteq} \{(F,A)_n\}$ implies $\{(H,C)_n\} \tilde{\subseteq} \{(F,A)_n\}^{\circ} \tilde{\subseteq} \{(F,A)_n\}$. Since $\{(F,A)_n\} \tilde{\subseteq} \{(G,B)_n\}$, $\{(H,C)_n\} \tilde{\subseteq} \{(F,A)_n\}^{\circ} \tilde{\subseteq} \{(F,A)_n\} \tilde{\subseteq} \{(G,B)_n\}$. This implies $\{(H,C)_n\} \tilde{\subseteq} \{(G,B)_n\}$. Since $\{(G,B)_n\}^{\circ} \tilde{\subseteq} \{(G,B)_n\}$, $\{(H,C)_n\} \tilde{\subseteq} \{(G,B)_n\}^{\circ}$. This implies $\{(F,A)_n\}^{\circ} \tilde{\subseteq} \{(G,B)_n\}^{\circ}$. This proves (iv).

v. By Definition 3.1 and by using (ii), $\{(F,A)_n\}^{\circ} \tilde{\subseteq} \{(F,A)_n\}$ and $\{(G,B)_n\}^{\circ} \tilde{\subseteq} \{(G,B)_n\}$. Using Remark 3.2, $\{(F,A)_n\}^{\circ}$ and $\{(G,B)_n\}^{\circ}$ is a fuzzy soft sequential open set implies $\{(F,A)_n\}^{\circ} \tilde{\cap} \{(G,B)_n\}^{\circ}$ is a fuzzy soft sequential open set. Therefore $\{(F,A)_n\}^{\circ} \tilde{\cap} \{(G,B)_n\}^{\circ} \tilde{\subseteq} \{(F,A)_n\} \tilde{\cap} \{(G,B)_n\}$. Since $(\{(F,A)_n\} \tilde{\cap} \{(G,B)_n\})^{\circ}$ is the largest fuzzy soft sequential open set contained in $\{(F,A)_n\} \tilde{\cap} \{(G,B)_n\}$, $(\{(F,A)_n\} \tilde{\cap} \{(G,B)_n\})^{\circ} \tilde{\subseteq} \{(F,A)_n\} \tilde{\cap} \{(G,B)_n\}$. This implies $\{(F,A)_n\}^{\circ} \tilde{\cap} \{(G,B)_n\}^{\circ} \tilde{\subseteq} (\{(F,A)_n\} \tilde{\cap} \{(G,B)_n\})^{\circ}$ ... **[Eqn. 3.5]**

On the other hand, we know that $\{(F,A)_n\} \tilde{\cap} \{(G,B)_n\} \tilde{\subseteq} \{(F,A)_n\}$ and $\{(F,A)_n\} \tilde{\cap} \{(G,B)_n\} \tilde{\subseteq} \{(G,B)_n\}$. Then by using (iv), $(\{(F,A)_n\} \tilde{\cap} \{(G,B)_n\})^{\circ} \tilde{\subseteq} \{(F,A)_n\}^{\circ}$ and $(\{(F,A)_n\} \tilde{\cap} \{(G,B)_n\})^{\circ} \tilde{\subseteq} \{(G,B)_n\}^{\circ}$. Hence $(\{(F,A)_n\} \tilde{\cap} \{(G,B)_n\})^{\circ} \tilde{\subseteq} \{(F,A)_n\}^{\circ} \tilde{\cap} \{(G,B)_n\}^{\circ}$ … **[Eqn. 3.6]**
By Eqn. 3.5 and Eqn. 3.6, (v) has proved.

vi. We know that, $\{(F,A)_n\} \tilde{\subseteq} \{(F,A)_n\} \tilde{\cup} \{(G,B)_n\}$ and $\{(G,B)_n\} \tilde{\subseteq} \{(F,A)_n\} \tilde{\cup} \{(G,B)_n\}$. Then by using (iv), $\{(F,A)_n\}^{\circ} \tilde{\subseteq} (\{(F,A)_n\} \tilde{\cup} \{(G,B)_n\})^{\circ}$ and $\{(G,B)_n\}^{\circ} \tilde{\subseteq} (\{(F,A)_n\} \tilde{\cup} \{(G,B)_n\})^{\circ}$ implies $\{(F,A)_n\}^{\circ} \tilde{\cup} \{(G,B)_n\}^{\circ} \tilde{\subseteq} (\{(F,A)_n\} \tilde{\cup} \{(G,B)_n\})^{\circ}$. This proves (vi).

Remark 3.6. The converse of (ii) and (vi) in Theorem 3.5 is need not be true. That is,

i. $\{(F,A)_n\} \tilde{\not\subseteq} \{(F,A)_n\}^{\circ}$
ii. $(\{(F,A)_n\} \tilde{\cup} \{(G,B)_n\})^{\circ} \tilde{\not\subseteq} \{(F,A)_n\}^{\circ} \tilde{\cup} \{(G,B)_n\}^{\circ}$

Now we will discuss the concept of closure in fuzzy soft sequential set.

Definition 3.7. Let $(U, \tilde{\tau}, E)$ be a fuzzy soft sequential topological space and $\{(F,A)_n\}$ be a fuzzy soft sequential set over U. Then the intersection of all fuzzy soft sequential closed sets over U containing $\{(F,A)_n\}$ is called the **fuzzy soft sequential closure** of $\{(F,A)_n\}$.

In other words, The **fuzzy soft sequential closure** of $\{(F,A)_n\}$ is denoted by $\overline{\{(F,A)_n\}} = \tilde{\cap}\{\{(G,B)_n\} : \{(F,A)_n\} \tilde{\subseteq} \{(G,B)_n\}, \{(G,B)^c_n\} \in \tilde{\tau}\}$.

Remark 3.8. Since the intersection of all fuzzy soft sequential closed sets is a fuzzy soft sequential closed set, then the fuzzy soft sequential closure is a fuzzy soft sequential closed set.

The example for the definition of fuzzy soft sequential closure is given below.

Example 3.9. Let us consider the universe $U = \{x_1, x_2\}$ and $A, B = \{e_1, e_2\} \subseteq E$ where $\{e_1, e_2\}$ be a collection of sets of parameters. Take $\tilde{\tau} = \{0^{\mathbb{N}}, 1^{\mathbb{N}}, \{(F,A)_n\}\}$ is a fuzzy soft sequential topological space.

Here $\{(F,A)_n\} = \left\{ \left(e_1, \left\{\frac{x_1}{(0.7)}, \frac{x_2}{(1-((1/2n))}\right\}\right), \left(e_2, \left\{\frac{x_1}{(0.3)}, \frac{x_2}{(3/4n)}\right\}\right) \right\} \ \forall\, n \in \mathbb{N}$

Then the corresponding fuzzy soft sequential closed sets are $0^{\mathbb{N}}, 1^{\mathbb{N}}, \{(F,A)^c_n\}$.

Consider $\{(J,A)_n\}$ be a fuzzy soft sequential set is defined as follows:

$\{(J,A)_n\} = \left\{ \left(e_1, \left\{\frac{x_1}{(1/3n)}, \frac{x_2}{(1/2n)}\right\}\right), \left(e_2, \left\{\frac{x_1}{(2/3n)}, \frac{x_2}{(1/5n)}\right\}\right) \right\} \ \forall\, n \in \mathbb{N}$

Then $\overline{\{(J,A)_n\}} = \{\{(J,A)_n\} \tilde{\subseteq} (1^{\mathbb{N}} \tilde{\cap} \{(F,A)^c_n\})\} = \{(F,A)^c_n\}$.

Theorem 3.10. Let $(U, \tilde{\tau}, E)$ be a fuzzy soft sequential topological space and $\{(F,A)_n\}$ be a fuzzy soft sequential set over U. Then

i. $\overline{\{(F,A)_n\}}$ is the smallest fuzzy soft sequential closed set containing $\{(F,A)_n\}$.
ii. $\{(F,A)_n\}$ is a fuzzy soft sequential closed set if and only if $\{(F,A)_n\} = \overline{\{(F,A)_n\}}$.

Proof.

i. Being the intersection of all fuzzy soft sequential closed sets, $\overline{\{(F,A)_n\}}$ is a fuzzy soft sequential closed set and contained in every fuzzy soft sequential closed set containing $\{(F,A)_n\}$. Hence $\overline{\{(F,A)_n\}}$ is the smallest fuzzy soft sequential closed set containing $\{(F,A)_n\}$.

ii. **Necessity:** Let $\{(F,A)_n\}$ be a fuzzy soft sequential closed set. Then $\{(F,A)_n\} \tilde{\subseteq} \{(F,A)_n\}$ and by using (i), $\overline{\{(F,A)_n\}}$ is the smallest fuzzy soft sequential closed set containing $\{(F,A)_n\}$. Therefore $\overline{\{(F,A)_n\}} \tilde{\subseteq} \{(F,A)_n\}$. By the definition of fuzzy soft sequential closure itself, $\{(F,A)_n\} \tilde{\subseteq} \overline{\{(F,A)_n\}}$. Hence $\{(F,A)_n\} = \overline{\{(F,A)_n\}}$.

Sufficiency: Suppose $\{(F,A)_n\} = \overline{\{(F,A)_n\}}$. By Remark 3.8, $\overline{\{(F,A)_n\}}$ is a fuzzy soft sequential closed set. Thus $\{(F,A)_n\}$ is a fuzzy soft sequential closed set.

Proposition 3.11. Let $(U,\tilde{\tau},E)$ be a fuzzy soft sequential topological space. If for a fuzzy soft sequential set $\{(F,A)_n\}$ over U, $\overline{\{(F,A)_n\}} = \{\overline{(F,A)_n}, n \in \mathbb{N}\}$, where $\overline{(F,A)_n}$ is the closure of $(F,A)_n$ in $(X_\gamma,\tilde{\tau})$.

Proof. For each $n \in \mathbb{N}$, $(F,A)_n$ is contained in the n^{th} component of $\overline{\{(F,A)_n\}}$ and hence the result.

Theorem 3.12. Let $(U,\tilde{\tau},E)$ be a fuzzy soft sequential topological space and $\{(F,A)_n\}, \{(G,B)_n\}$ be the fuzzy soft sequential sets over U. Then the following properties are hold:

i. $\overline{0^{\mathbb{N}}} = 0^{\mathbb{N}}$, $\overline{1^{\mathbb{N}}} = 1^{\mathbb{N}}$
ii. $\{(F,A)_n\} \tilde{\subseteq} \overline{\{(F,A)_n\}}$
iii. $\overline{\overline{\{(F,A)_n\}}} = \overline{\{(F,A)_n\}}$
iv. If $\{(F,A)_n\} \tilde{\subseteq} \{(G,B)_n\}$, then $\overline{\{(F,A)_n\}} \tilde{\subseteq} \overline{\{(G,B)_n\}}$
v. $\overline{(\{(F,A)_n\} \tilde{\cap} \{(G,B)_n\})} \tilde{\subseteq} \overline{\{(F,A)_n\}} \tilde{\cap} \overline{\{(G,B)_n\}}$
vi. $\overline{(\{(F,A)_n\} \tilde{\cup} \{(G,B)_n\})} = \overline{\{(F,A)_n\}} \tilde{\cup} \overline{\{(G,B)_n\}}$

Proof:

i. Clearly, intersection of all fuzzy soft sequential closed sets containing $0^{\mathbb{N}}$ and $1^{\mathbb{N}}$ is always $0^{\mathbb{N}}$ and $1^{\mathbb{N}}$. Hence proved (i).

ii. Using Definition 3.7, $\{(F,A)_n\} \tilde{\subseteq} \overline{\{(F,A)_n\}}$.

iii. Let $\overline{\{(F,A)_n\}} = \{(G,B)_n\}$. By using Remark 3.8, $\{(G,B)_n\}$ is a fuzzy soft sequential closed set. By using (ii) of Theorem 3.10, $\overline{\{(G,B)_n\}} = \{(G,B)_n\}$. This proves (iii).

iv. Let $\{(F,A)_n\} \tilde{\subseteq} \{(G,B)_n\}$. Let $\{(H,C)_n\} \tilde{\subseteq} \overline{\{(F,A)_n\}}$. From Definition 3.7 and by using (ii), $\{(F,A)_n\} \tilde{\subseteq} \overline{\{(F,A)_n\}}$ implies $\{(H,C)_n\} \tilde{\subseteq} \{(F,A)_n\} \tilde{\subseteq} \overline{\{(F,A)_n\}}$. Since $\{(F,A)_n\} \tilde{\subseteq} \{(G,B)_n\}, \{(H,C)_n\} \tilde{\subseteq} \{(G,B)_n\}$. Since $\{(G,B)_n\} \tilde{\subseteq} \overline{\{(G,B)_n\}}$, $\{(H,C)_n\} \tilde{\subseteq} \overline{\{(G,B)_n\}}$. Hence $\overline{\{(F,A)_n\}} \tilde{\subseteq} \overline{\{(G,B)_n\}}$.

v. We know that, $\{(F,A)_n\} \tilde{\cap} \{(G,B)_n\} \tilde{\subseteq} \{(F,A)_n\}$and $\{(F,A)_n\} \tilde{\cap} \{(G,B)_n\} \tilde{\subseteq} \{(G,B)_n\}$. By using (iv), $\overline{(\{(F,A)_n\} \tilde{\cap} \{(G,B)_n\})} \tilde{\subseteq} \overline{\{(F,A)_n\}}$ and $\overline{(\{(F,A)_n\} \tilde{\cap} \{(G,B)_n\})} \tilde{\subseteq} \overline{\{(G,B)_n\}}$. This implies $\overline{(\{(F,A)_n\} \tilde{\cap} \{(G,B)_n\})} \tilde{\subseteq} \overline{\{(F,A)_n\}} \tilde{\cap} \overline{\{(G,B)_n\}}$. This proves (v).

vi. From Definition 3.7 and by (ii), $\{(F,A)_n\} \tilde{\subseteq} \overline{\{(F,A)_n\}}$ and $\{(G,B)_n\} \tilde{\subseteq} \overline{\{(G,B)_n\}}$. Then$\{(F,A)_n\} \tilde{\cup} \{(G,B)_n\} \tilde{\subseteq} \overline{\{(F,A)_n\}} \tilde{\cup} \overline{\{(G,B)_n\}}$. Since $\overline{(\{(F,A)_n\} \tilde{\cup} \{(G,B)_n\})}$ is the smallest fuzzy soft sequential closed set containing $\{(F,A)_n\} \tilde{\cup} \{(G,B)_n\}$, hence

$$\overline{(\{(F,A)_n\} \tilde{\cup} \{(G,B)_n\})} \tilde{\subseteq} \overline{\{(F,A)_n\}} \tilde{\cup} \overline{\{(G,B)_n\}} \qquad \ldots \textbf{[Eqn. 3.7]}$$

Conversely, $\{(F,A)_n\} \tilde{\subseteq} \{(F,A)_n\} \tilde{\cup} \{(G,B)_n\}$ and $\{(G,B)_n\} \tilde{\subseteq} \{(F,A)_n\} \tilde{\cup} \{(G,B)_n\}$.
By(iv), $\overline{\{(F,A)_n\}} \tilde{\subseteq} \overline{(\{(F,A)_n\} \tilde{\cup} \{(G,B)_n\})}$ and $\overline{\{(G,B)_n\}} \tilde{\subseteq} \overline{(\{(F,A)_n\} \tilde{\cup} \{(G,B)_n\})}$
Therefore $\overline{\{(F,A)_n\}} \tilde{\cup} \overline{\{(G,B)_n\}} \tilde{\subseteq} \overline{(\{(F,A)_n\} \tilde{\cup} \{(G,B)_n\})}$ … **[Eqn. 3.8]**
From Eqn. 3.7 and Eqn. 3.8, (vi) has proved.

Remark 3.13. The converse of (ii) and (v) in Theorem 3.12 is need not be true. That is,

i. $\overline{\{(F,A)_n\}} \tilde{\not\subset} \{(F,A)_n\}$
ii. $\overline{\{(F,A)_n\}} \tilde{\cap} \overline{\{(G,B)_n\}} \tilde{\not\subset} \overline{(\{(F,A)_n\} \tilde{\cap} \{(G,B)_n\})}$

Proposition 3.14. Let $(U,\tilde{\tau},E)$ be a fuzzy soft sequential topological space and $\{(F,A)_n\}$ be a fuzzy soft sequential set over U. Then the following properties are hold:

i. $\{(F,A)_n\}^{\circ} = (\overline{\{(F,A)_n^c\}})^c$
ii. $\overline{\{(F,A)_n\}} = (\{(F,A)_n^c\}^{\circ})^c$
iii. $(\overline{\{(F,A)_n\}})^c = \{(F,A)_n^c\}^{\circ}$
iv. $\overline{\{(F,A)_n^c\}} = (\{(F,A)_n\}^{\circ})^c$

Proof:

i. $(\overline{\{(F,A)_n^c\}})^c = (\tilde{\cap}\{\{(G,B)_n\} : \{(F,A)_n^c\} \tilde{\subseteq} \{(G,B)_n\}, \{(G,B)_n\} \in \tilde{\tau}^c\})^c$
$= \tilde{\cup}\{\{(G,B)_n^c\} : \{(F,A)_n^c\}^c \tilde{\supseteq} \{(G,B)_n^c\}, \{(G,B)_n^c\} \in \tilde{\tau}\}$
$= \tilde{\cup}\{\{(G,B)_n^c\} : \{(F,A)_n\} \tilde{\supseteq} \{(G,B)_n^c\}, \{(G,B)_n^c\} \in \tilde{\tau}\}$
$= \{(F,A)_n\}^{\circ}$

ii. $(\{(F,A)_n^c\}^{\circ})^c = (\tilde{\cup}\{\{(G,B)_n\} : \{(G,B)_n\} \tilde{\subseteq} \{(F,A)_n^c\}, \{(G,B)_n\} \in \tilde{\tau}\})^c$
$= \tilde{\cap}\{\{(G,B)_n^c\} : \{(G,B)_n^c\} \tilde{\supseteq} \{(F,A)_n^c\}^c, \{(G,B)_n^c\} \in \tilde{\tau}^c\}$
$= \tilde{\cap}\{\{(G,B)_n^c\} : \{(G,B)_n^c\} \tilde{\supseteq} \{(F,A)_n\}, \{(G,B)_n^c\} \in \tilde{\tau}^c\}$
$= \overline{\{(F,A)_n\}}$

iii. $(\overline{\{(F,A)_n\}})^c = (\tilde{\cap}\{\{(G,B)_n\} : \{(F,A)_n\} \tilde{\subseteq} \{(G,B)_n\}, \{(G,B)_n\} \in \tilde{\tau}^c\})^c$
$= \tilde{\cup}\{\{(G,B)_n^c\} : \{(F,A)_n^c\} \tilde{\supseteq} \{(G,B)_n^c\}, \{(G,B)_n^c\} \in \tilde{\tau}\}$
$= \{(F,A)_n^c\}^{\circ}$

iv. $(\{(F,A)_n\}^{\circ})^c = (\tilde{\cup}\{\{(G,B)_n\} : \{(G,B)_n\} \tilde{\subseteq} \{(F,A)_n\}, \{(G,B)_n\} \in \tilde{\tau}\})^c$
$= \tilde{\cap}\{\{(G,B)_n^c\} : \{(G,B)_n^c\} \tilde{\supseteq} \{(F,A)_n^c\}, \{(G,B)_n^c\} \in \tilde{\tau}^c\}$
$= \overline{\{(F,A)_n^c\}}$

4. Conclusion In this paper, we have introduced and discussed the notion of fuzzy soft sequential interior using fuzzy soft sequential open sets. Moreover, the concept of fuzzy soft sequential closure in fuzzy soft sequential topological spaces are proposed. We have explored the features, theorems, obtained some results with needed examples. Finally, we have presented the properties related to the notion of complement using the fuzzy soft sequential interior and fuzzy soft sequential closure.

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