

γ_D – POLYNOMIAL OF SOME STANDARD GRAPHS

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Abstract: The notion of γ –polynomial was introduced by S. Alikhani and Y. H. Peng [1]. Here we introduce that γ –polynomial in detour number of graph. The definition of γ_d –polynomial of G is the polynomial $\gamma_D(G, x) = \sum \gamma_d(G, i)x^i$ where $\gamma_d(G, i)$ is the number of γ_d –set of G of cardinality i . In this paper, we compute for some standard graphs.

Keywords:- detour number, detour set, domination polynomial, detour domination polynomial.

Introduction

The domination concept was introduced by Berge and Ore in 1962 [5] is currently receiving much attention in literature. For basic definitions and terminologies of graphs we refer to [1,5].

Definition 1.1[4]: A set $D \subseteq V(G)$ said to be a γ_d –set of G if the set is the minimum of among all the sets having below conditions:

- (i) The vertices of G lie on detour path of some pair of vertices of D .
- (ii) The vertices of G is either in D or adjacent to a vertex of D .

Definition 1.2[1]: The d –polynomial of graph G is the polynomial $D(G, x) = \sum d(G, i)x^i$ where $d(G, i)$ is the number of dominating sets of G of cardinality i

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Definition 2.1: The γ_d –polynomial of G is the polynomial $\gamma_D(G, x) = \sum \gamma_d(G, i)x^i$ where $\gamma_d(G, i)$ is the number of γ_d –set of G of cardinality i .

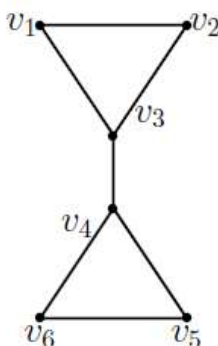


Figure 2.1

Example 2.2: The graph G in figure 2.1, $\gamma_d(G, 2) = 4, \gamma_d(G, 3) = 12, \gamma_d(G, 4) = 11, \gamma_d(G, 5) = 6$, and $\gamma_d(G, 6) = 1$. Therefore, $\gamma_D(G, x) = 4x^2 + 12x^3 + 11x^4 + 6x^5 + x^6$.

Theorem 2.3: The complete graph K_n , $\gamma_D(K_n, x) = \sum^n C_i x^i$ where $2 \leq i \leq n$.

Proof: Let K_n be a complete graph. By the definition of d -set, any two vertices form the γ_d -set of K_n . Therefore, the number of γ_d -sets with cardinality 2 is ${}^nC_2$. Clearly, any set of cardinality greater than or equal to the $\gamma_d(G)$ is form γ_d -set of G . Therefore, the number of γ_d -sets with cardinality i is ${}^nC_i$. Hence, the co-efficient of x^i is ${}^nC_i$.

Hence, $\gamma_D(K_n, x) = \sum^n C_i x^i$ where $2 \leq i \leq n$.

Theorem 2.4: The Wheel graph W_n , $\gamma_D(W_n, x) = \sum^n C_{i-1} x^i + 2x^n + x^{n+1}$ where $2 \leq i \leq n-1$.

proof: Let W_n be a wheel graph and let u be a central vertex of W_n . Since u is the central vertex of W_n , u dominates all the vertices of W_n . Therefore, the central vertex u with any vertex of W_n forms the γ_d -set of G . Hence $\gamma_d(W_n, 2) = {}^nC_1$. Clearly, any set of cardinality greater than or equal to 2 containing u forms the γ_d -set of G . Hence, $\gamma_d(G, i) = {}^nC_{i-1}$. Therefore, $\gamma_D(W_n, x) = \sum^n C_{i-1} x^i + 2x^n + x^{n+1}$ where $2 \leq i \leq n-1$.

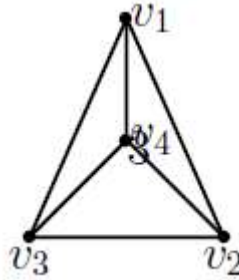


Figure 2.2

Example 2.5: The wheel graph W_3 in figure 2.2, $\gamma_d(W_n, 2) = 3, \gamma_d(W_n, 3) = 3$, and $\gamma_d(W_n, 4) = 1$. Therefore, $\gamma_D(W_n, x) = 3x^2 + 3x^3 + x^4$.

Remark 2.6: Any subset of $V(G)$ of cardinality is greater than or equal to $\gamma_d(G)$ is need not be a γ_d -set of graph G . For the friendship graph F_2 in figure 2.3, the set $\{v_1, v_5\}$ is a γ_d -set of G but $\{v_3, v_4, v_5\}$ is not a γ_d -set of F_2 .

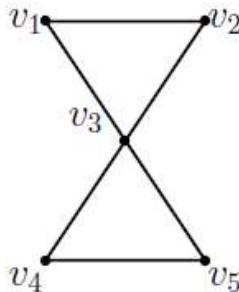


Figure 2.3

Remark 2.7:

1. Suppose S is a γ_d -set of G but $V - S$ is need not be a γ_d -set of G . For the path P_7 in figure 2.4, the set $S = \{v_1, v_4, v_7\}$ is the γ_d -set of G but $V - S = \{v_2, v_3, v_5, v_6\}$ is not a

γ_d –set of G . Any subset of $V(G)$ containing γ_d –set of G is γ_d –set of G . The converse is not true.

- For the path P_7 in figure 2.4, the set $S = \{v_1, v_4, v_7\}$ is the γ_d –set of G but the set $\{v_1, v_3, v_5, v_7\}$ is also γ_d –set of G and S is not contained in $\{v_1, v_3, v_5, v_7\}$.

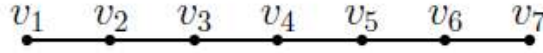


Figure 2.4

Remark 2.9:

- Let $x \in G$ such that $\Delta(G) = \deg x$. Then x need not be in the γ_d –set. For the graph G in figure 2.5, the set $\{u, v\}$ is the γ_d –set and $\Delta(G) = \deg x = 3$ but $x \notin \gamma_d$ –set.
- The co-efficient of $x^{\gamma_d(G)}$ in detour domination polynomial need not be 1. For example 2.7, the $\gamma_D(W_4, x) = 3x^2 + 2x^3 + x^4$.

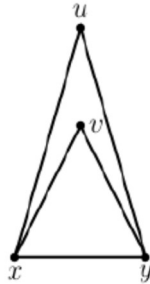


Figure 2.5

Observation 2.10: The Co-efficient of x^n in the γ_d –polynomial $\gamma_D(G, x)$ is always 1.

Theorem 2.11: If the set of end vertices E is γ_d –set of graph G , then the co-efficient of $x^{\gamma_d(G)}$ is 1.

Proof: Suppose the set of end vertices E is the γ_d – set of graph G .

To Prove the co-efficient of $x^{\gamma_d(G)}$ is 1. i.e. to prove $g(G, \gamma_d(G)) = 1$. Suppose not, $g(G, \gamma_d(G)) \neq 1$. W.L.G., $\gamma_D(G, \gamma_d(G)) = 2$. Therefore, F is also a γ_d –set of G . By our assumption, E and F are two distinct sets.

Therefore, $\exists$ element z s.t. $z \in E$ and $z \notin F$.

Since $z \in E$, z is an end vertices of G . Therefore by the definition of γ_d –set, neither z is in any pair of detour path of F nor $x \in T$.

Therefore, F is not a γ_d – set of graph G which is contradiction to our assumption. Hence the co-efficient of $x^{\gamma_d(G)}$ is 1.

Theorem 2.12: If the set of end vertices E is γ_d –set of graph G , then E is unique.

Proof: Proof is obvious.

Theorem 2.13: If the set of end vertices E is γ_d –set of graph G ,

then $\gamma_D(G, x) = \sum_{i=0}^{|V-E|} \cdot |V-E| C_i x^{|E|+i}$.

Proof: Suppose the set of end vertices E is γ_d –set of graph G .

By above theorem, E is the only γ_d –set of G .

Case(i) Suppose, $|V - E| = 0$. Obviously, $\gamma_D(G, x) = x^{|S|} = x^n$.

Case(ii) Suppose, $|V - E| \neq 0$. Let $V - S = \{v_1, v_2, \dots, v_{|V-E|}\}$. Consider, the set $E \cup \{v_i\}$ is γ_d -set with cardinality $|E| + 1$. Hence, $E \cup \{v_i\}_{i=1}^{|V-E|}$ are the only γ_d -set with cardinality $|E| + 1$. Similarly, the set $S \cup \{v_i, v_j\}_{i \neq j}^{|V-S|}$ is γ_d -set with cardinality $|S| + 2$.

Proceeding like this, we get $\gamma_D(G, x) = \sum_{i=0}^{|V-E|} C_i x^{|E|+i}$.

Remark 2.14: The converse is need not be true by the graph complete graph K_n for $n \geq 3$.

Theorem 2.15: The γ_d -polynomial for some standard graphs:

1. For a bi-star graph $B(2, n)$, $\gamma_D(B(2, n), x) = x^{n-2} + 2x^{n-1} + x^n$.
2. For a book graph B_n , $\gamma_D(B_n, x) = 2 \sum_{i=0}^{|V-S|} C_i x^{|S|+i}$.

Proof: The proof of 1 obvious by theorem.

2. Let $V(B_n) = \{x, y, u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$ and $E(B_n) = \{xv_i, yu_i, u_i v_i\}$ be the vertices and edges of B_n . Consider, the set $S_1 = \{y, v_i\}$ and $S_2 = \{x, u_i\}$ which are the γ_d -set of B_n .

Now consider the set $V - S_1 = \{v_1, v_2, \dots, v_{|V-S_1|}\}$.

Clearly, the set $S_1 \cup \{v_i\}$ are γ_d -set with cardinality $|S_1| + 1$. The set $S_1 \cup \{v_i, v_j\}$ ($i \neq j$) is γ_d -set. In this similar way, S_1 with any distinct k vertices of $V - S_1$. Therefore, $2 \sum_{i=0}^{|V-S_1|} C_i x^{|S_1|+i}$.

On the other hand, S_1 is replaced by S_2 in above, we get $2 \sum_{i=0}^{|V-S_2|} C_i x^{|S_2|+i}$.

Hence, adding both we get $\gamma_D(B_n, x) = 2 \sum_{i=0}^{|V-S|} C_i x^{|S|+i}$.

Theorem 2.16: For a Octopus graph O_n , $\gamma_D(O_n, x) = x^{n-1} + x^n$.

Proof: Let $V(O_n) = \{u, v_1, v_2, \dots, v_n, u_1, u_2, \dots, u_n\}$ be the vertices of Octopus graph O_n , where the set $\{v_1, v_2, \dots, v_n\}$ is the vertices of F_n and the set $\{u, u_1, u_2, \dots, u_n\}$ is the vertices of $K_{1,n}$. By definition of detour set, the set of vertices $\{u_1, u_2, \dots, u_n\}$ is lie in γ_d -set of G . Clearly, the set $\{v_1, v_2, \dots, v_n\}$ which is also in γ_d -set of G . Therefore, $\{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$ is the only γ_d -set of G . Hence, the γ_d -polynomial of O_n is $x^{n-1} + x^n$.

Theorem 2.17: If S is the γ_d -set of G and if the co-efficient of $x^{|S|+1}$ is equal to $|V - S|$, then S is unique.

Proof: Let S be γ_d -set of G . Suppose the co-efficient of $x^{|S|+1}$ is equal to $|V - S|$. To prove S is unique. Suppose not, S_1 and S_2 is γ_d -set of G . Hence, the set $S_1 \cup \{v_i\}$ ($v_i \in V - S$) is the γ_d -set with cardinality $|S| + 1$.

In the similar way, $S_2 \cup \{u_i\}$ ($u_i \in V - S$) is the γ_d -set with cardinality $|S| + 1$. Hence, the co-efficient of $x^{|S|+1}$ is equal to $2|V - S|$ which is contradiction to our assumption. Therefore, If the co-efficient of $x^{|S|+1}$ is equal to $|V - S|$, then S is unique.

Remark 2.18: The converse is not true by the graph G in figure 2.6, the set $E = \{l, o\}$ is γ_d -set which is unique and $|V - S| = 4$ but the sets $\{l, n, o\}, \{l, o, p\}, \{l, m, o\}, \{l, o, q\}, \{l, n, p\}$ are γ_d -set with cardinality 3. Therefore the co-efficient of $x^{|S|+1}$ is 5.

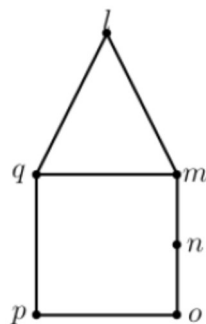


Figure 2.6

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